MATH 11 FALL 2008: LECTURE 11

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1. Announcements

(1) Midterm Friday

2. The exponential function

Recall, we have defined e to be the number that satisfies $\ln e = 1$.

Proposition 1.

$$2 < e < 4.$$

Proof. *To see $e > 2$, note that*

$$\int_1^2 \frac{1}{t} dt$$

is less than the area of the unit square. To see that $4 > e$, note that

$$\int_1^4 \frac{1}{t} dt = \int_1^2 \frac{1}{t} dt + \int_1^2 \frac{1}{t} dt > \int_1^2 \frac{1}{2} dt + \int_2^4 \frac{1}{4} dt = \frac{1}{2} + \frac{1}{2} = 1.$$

□

2.1. Inverse functions.

Suppose that f is a real function such that every real number y is hit exactly once by $f(x)$. Then f has an *inverse function* g defined as follows. Let x be a number. Then there is some other real number a such that $f(a) = x$. Define

$$g(x) = a.$$

In other words, we can define the set $f^{-1}(x)$ as

$$f^{-1}(x) = \{a \in \mathbb{R} : f(a) = x\}.$$

Then f has an inverse function if $f^{-1}(x)$ consists of exactly one element for each real number x.

Definition 2. *The function g is an* inverse function *of f if $f(g(x)) = x$ and $g(f(x)) = x$ for all x in the domain of g and f, respectively.*

Example 3. *Let $f(x) = 5x$. Then let y be a real number. Then there is exactly one x such that $f(x) = y$, namely, let $x = y/5$. Therefore f has an inverse function. Let $g(x) = x/5$. Then*

$$f(g(x)) = f(x/5) = 5 \cdot x/5 = x,$$

and

$$g(f(x)) = g(5x) = (5x)/5 = x.$$

Therefore g is indeed the inverse function of f.

Date: September 23, 2008.

Remark 4. Note that in the above, we solved for x in terms of y to discover the inverse function. This is encoded in the geometry of f and f^{-1} in that the graph of f^{-1} is obtained by reflecting the graph of f through the line $y = x$.

2.2. **The inverse of the natural logarithm.** Last time we discovered that every real number y is hit by exactly once by $\ln x$. Thus, $\ln x$ has an inverse function.

Definition 5. *The* exponential function *is the inverse of the natural logarithm, and is denoted* e^x.

So, by definition we have for all real numbers x

$$\ln(e^x) = x,$$

and for all $x > 0$,

$$e^{(\ln x)} = x.$$

Theorem 6.

$$\frac{d}{dx}(e^x) = e^x.$$

Proof. We use implicit differentiation. For all x we have

$$\begin{aligned} \ln(e^x) &= x \\ \iff \frac{1}{e^x}\frac{d}{dx}(e^x) &= 1 \\ \iff \frac{d}{dx}(e^x) &= e^x \end{aligned}$$

□

Theorem 7.

$$e^{x+y} = e^x \cdot e^y$$

Proof. Note that $x = \ln(e^x)$ and $y = \ln(e^y)$. Thus

$$e^{(x+y)} = e^{(\ln e^x + \ln(e^y))} = e^{\ln(e^x e^y)} = e^x e^y.$$

3. General exponential functions

We can define numbers like $2^{\sqrt{2}}$ using the exponential function. (How could we do so otherwise?)

Definition 8. *For any* $a > 0$ *and* $x \in \mathbb{R}$,

$$a^x = e^{x \ln a}.$$

Proposition 9.

$$a^{x+y} = a^x a^y.$$

Proof.
$$a^{x+y} = e^{(x+y)\ln a} = e^{x\ln a + y\ln a} = e^{x\ln a}e^{y\ln a} = a^x a^y.$$

We can also use this to prove a general power rule. By definition,
$$x^r = e^{r\ln x},$$
so
$$\begin{aligned}\frac{d}{dx}(x^r) &= \frac{d}{dx}(e^{r\ln x}) \\ &= e^{r\ln x}\cdot\frac{d}{dx}(r\ln x) \\ &= x^r\cdot r\cdot\frac{1}{x} \\ &= rx^{r-1}.\end{aligned}$$

Corollary 10. *For any real number r,*
$$\ln(x^r) = r\ln x.$$

Theorem 11.
$$(a^b)^c = a^{bc}$$

Proof.
$$(a^b)^c = e^{c\ln a^b} = e^{bc\ln a} = a^{bc}.$$

Theorem 12.
$$\frac{d}{dx}(a^x) = a^x\ln a.$$

Proof.
$$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x\ln a}) = e^{x\ln a}\ln a = a^x\ln a.$$

4. General logarithms

Definition 13. *For $a > 0$,*
$$\log_a x = \frac{\ln x}{\ln a}$$