Math 171 - Abstract Algebra I Section _
HW # 16
11/4/10

Dummit & Foote (8.3) 1, 2, 8, 11

8.3.1 Let $G = \mathbb{Q}^\times$ be the multiplicative group of nonzero rational numbers. If $\alpha = p/q \in G$, where p and q are relatively prime integers, let $\varphi : G \to G$ be the map which interchanges the primes 2 and 3 in the prime power factorizations of p and q (so, for example, $\varphi(2^4 3^{11} 5^1 13^2) = 3^4 2^{11} 5^1 13^2$, $\varphi(3/16) = \varphi(3/2^4) = 2/3^4 = 2/81$, and φ is the identity on all rational numbers with numerators and denominators relatively prime to 2 and to 3).

(a) Prove that φ is a group isomorphism.

(b) Prove that there are infinitely many isomorphisms of the group G to itself.

(c) Prove that none of the isomorphisms above can be extended to an isomorphism of the *ring* $\mathbb{Q}$ to itself. In fact prove that the identity map is the only ring isomorphism of $\mathbb{Q}$.

8.3.2 Let a and b be elements of the Unique Factorization Domain R. Prove that a and b have a least common multiple (cf. Exercise 11 of Section 1) and describe it in terms of the prime factorizations of a and b in the same fashion that Proposition 13 describes their greatest common divisor.

8.3.8 Let R be the quadratic integer ring $\mathbb{Z}[\sqrt{-5}]$ and define the ideals $I_2 = (2, 1+\sqrt{-5})$, $I_3 = (3, 2+\sqrt{-5})$, and $I_3' = (3, 2-\sqrt{-5})$.

(a) Prove that 2, 3, $1+\sqrt{-5}$ and $1-\sqrt{-5}$ are irreducibles in R, no two of which are associate in R, and that $6 = 2 \cdot 3 = (1+\sqrt{-5}) \cdot (1-\sqrt{-5})$ are two distinct factorizations of 6 into irreducibles in R.

(b) Prove that I_2, I_3, and I_3' are prime ideals in R. [One approach: for I_3, observe that $R/I_3 \cong (R/(3))/(I_3/(3))$ by the Third Isomorphism Theorem for Rings. Show that $R/(3)$ has 9 elements, $(I_3/(3))$ has 3 elements, and that $R/I_3 \cong \mathbb{Z}/3\mathbb{Z}$ as an additive abelian group. Conclude that I_3 is a maximal (hence prime) ideal and that $R/I_3 \cong \mathbb{Z}/3\mathbb{Z}$ as rings.]

(c) Show that the factorizations in (a) imply the equality of ideals $(6) = (2)(3)$ and $(6) = (1+\sqrt{-5})(1-\sqrt{-5})$. Show that these two ideal factorizations give the same factorization of the ideal (6) as the product of prime ideals (cf. Exercise 5 in Section 2).

8.3.11 (*Characterization of P.I.D.s*) Prove that R is a P.I.D. if and only if R is a U.F.D. that is also a Bezout Domain (cf. Exercise 7 in Section 2). [One direction is given by Theorem 14. For the converse, let a be a nonzero element of the ideal I with a minimal number of irreducible factors. Prove that $I = (a)$ by showing that if there is an element $b \in I$ that is not in (a) then $(a, b) = (d)$ leads to a contradiction.]