Abstract Algebra
Lecture 18
Monday, 11/8/2010

1 Category Theory

1.1 Classes

Big Idea: We've seen similar structures in Group Theory and Ring Theory so far this semester. Groups, rings, subgroups, subrings, quotient groups, quotient rings, sums of groups and rings, isomorphism theorems. Can we formalize the idea of having similar structures in different settings? Maybe even not only for groups and rings, but also things like "smooth manifolds", "topological spaces", vector spaces, etc.

We can describe relations between these different mathematical settings with **category theory**.

The first danger of category theory is the Barber's Paradox. Suppose there is a town where the barber cuts the hair of everyone who doesn't cut their own hair. So who cuts the barber's hair? This paradox is similar to the paradox that shows that there is no *set of all sets*. So we know we can't use set theory to describe these settings – we can't have a set of all groups, since groups are sets.

Defn: A **class** is a mathematical collection which is *not necessarily a set*. A **proper class** is a class which is not a set.

We just defined our way out of the Barber's Paradox – a class doesn't need to follow set theory axioms.

Ex: The class of all sets is a proper class.

1.2 Categories

Defn: A **category** $\mathcal{C}$ consists of

(1) a class whose elements are called the **objects** of $\mathcal{C}$,
(2) a second class whose elements are called **morphisms**,
(3) **two functions** which assign to every morphism α a domain and codomain, which are objects of $\mathcal{C}$,
(4) a **partially defined function** which assigns to some pairs (α, β) of morphisms the product or composition $\alpha\beta$.

such that

(a) $\alpha\beta$ is defined when $\text{dom}(\alpha) = \text{codom}(\beta)$. Then $\text{dom}(\alpha\beta) = \text{dom}(\beta)$ and $\text{codom}(\alpha\beta) = \text{codom}(\alpha)$.
(b) For all $A \in \text{Ob}(\mathcal{C})$, there exists an **identity morphism** $\mathbb{1}_A : A \to A$ such that: $\alpha\mathbb{1}_A = \alpha$ for all morphisms α such that $\text{dom}(\alpha) = A$, and $\mathbb{1}_A\beta = \beta$ for all β such that $\text{codom}(\beta) = A$.
(c) If $\alpha\beta$, $\beta\gamma$ are defined, then $\alpha(\beta\gamma) = (\alpha\beta)\gamma$ are defined.

Ex: The category $\mathcal{S}$ets has objects which are sets, morphisms are maps of sets.

Ex: The category $\mathcal{G}$roups has Ob($\mathcal{G}$) = groups, Mor($\mathcal{G}$) = group homomorphisms.

Ex: The category $\mathcal{R}$ings has Ob($\mathcal{R}$) = rings, and Mor($\mathcal{R}$) = ring homomorphisms.

Note: Note that each morhpism can only have on domain and one codomain. In sets, $\mathbb{1}_\mathbb{Z} : \mathbb{Z} \to \mathbb{Z}$, $\iota : \mathbb{Z} \hookrightarrow \mathbb{Q}$.

Defn: A category is **small** if its objects and morphisms form sets.

Remark: If Mor($\mathcal{C}$) is a set, then $\mathcal{C}$is small

Ex: A **preordered set** is a set together with a reflexive transitive relation $\leq$. Every preordered set I forms a category. $\text{Ob}(I)$: elements of I. The morphisms are ordered pairs (i, j) such that $i \leq j$, in other words $(i, j) : i \to j$. Also, $(j, k)(i, j) = (i, k)$.

Ex: A directed graph Γ is (1) a set $V(\Gamma)$ of vertices, (2) a set $E\Gamma$ of edges, which is also (3) two mappings assigning to each edge an origin and a destination.

Fact: every small category is a directed graph.
It the converse true? Yes, and the proof is in our homework.

Theorem: When Γ is a directed graph, the vertices and paths are the objects and morhpisms of a category $\hat{\Gamma}$.

Defn: $\hat{\Gamma}$ is the **free category** on Γ, or the **category of paths**.

1.3 Types of Morphisms

Defn: A **monomorphism** is a morphism μ such that $\mu\alpha = \mu\beta \Rightarrow \alpha = \beta$.

Defn: An **epimorphism** is a morphism σ such that $\alpha\sigma = \beta\alpha \Rightarrow \alpha = \beta$.

Defn: An **isomorphism** is a morphism $\alpha : A \to B$ having an inverse $\beta : B \to A$ such that $\alpha\beta = \mathbb{1}_B$ and $\beta\alpha = \mathbb{1}_A$.

Note: Note that monomorphism is in some ways similar to injectivity, and epimorphism is similar to surjectivity. In fact, if α is injective, then α is a monomorphism, and if β is surjective, then β is an epimorphism, though the converses are not quite always true.

Proof: Let μ be injective. Suppose $\mu\alpha = \mu\beta$ for some $\alpha, \beta : A \to B$ (so we also have $\mu : B \to C$). Then for al $a \in A$, we have $\mu(\alpha(a)) = \mu(\beta(a))$, so $\alpha(a) = \beta(a)$. Thus $\alpha = \beta$. etc. ∎