LECTURE 1: LINES IN $\mathbb{R}^3$

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ABSTRACT. In this first lecture, we'll review some necessary material and notation from linear algebra, and begin to explore geometry in $\mathbb{R}^3$. In particular, we'll study lines in 3-space.

1. REVIEW: LINEAR ALGEBRA

Since we'll be working in $\mathbb{R}^n$, where $n > 1$, it's useful to have the language of linear algebra at hand. Since we've all seen this before, let's quickly review and standardize notation.

Definition 1. *We define n-dimensional real space as follows.*

$$\mathbb{R}^n = \{(x_1, \ldots, x_n) : x_i \in \mathbb{R}\}$$
$$\mathbb{R} = \textit{ The set of real numbers}$$

It is important to note that we may think of elements of $\mathbb{R}^n$ either as points or *vectors*. Similarly, we may think of elements of $\mathbb{R}$ as numbers or as *scalars*. Let's keep both of these viewpoints in mind during our explorations; they each may come in handy.

With our vector viewpoint, we are pointing out that $\mathbb{R}^n$ is in fact a real vector space. Hooray! Vectors have lots of nice properties. In particular, we have addition and scalar multiplication.

Definition 2. *Let $\vec{a} = (a_1, \ldots, a_n)$, $\vec{b} = (b_1, \ldots, b_n)$ be vectors in $\mathbb{R}^n$. The (vector) addition of $\vec{a}$ and $\vec{b}$ is defined by*

$$\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n).$$

For any real number $k \in \mathbb{R}$, the scalar multiple of $\vec{a}$ by k is

$$k\vec{a} = (ka_1, \ldots, ka_n).$$

These are fine and dandy algebraic formulations of addition and scalar multiplication in n-space. But what the heck is going on geometrically? Can we find a geometric interpretation of addition and scalar multiplication? Yes indeed!

Date: March 9, 2009.

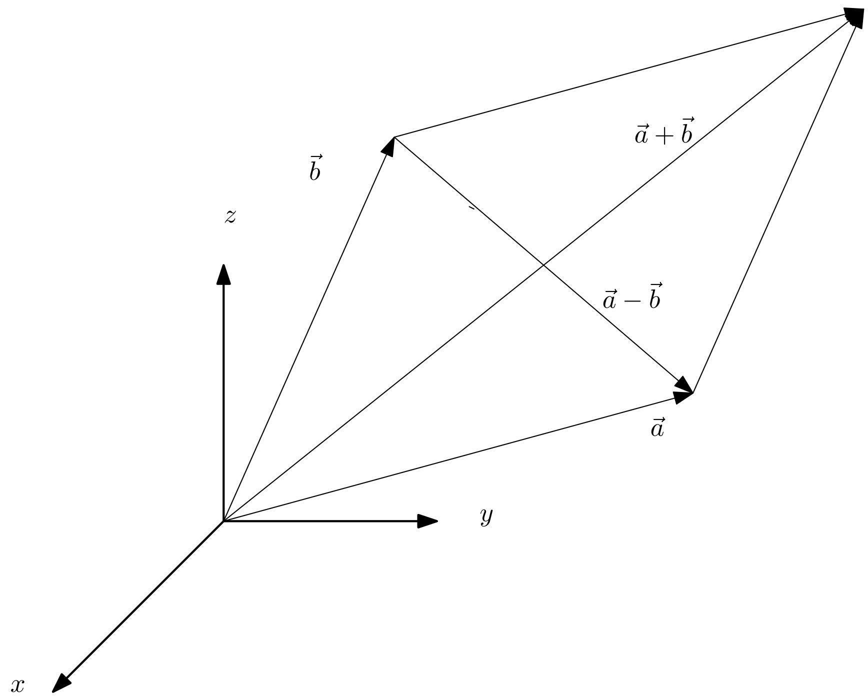

FIGURE 1. Geometric interpretation of vector addition (and subtraction)

1.1. **Basic Geometry of vectors.** Before we can discuss the geometry of a sum of vectors, let's recall the geometric interpretation of a single vector. Inspired by physical arguments, we may regard a vector as a *directed line segment* or a line segment with both *magnitude (length) and direction*.

How so? As follows.

Definition 3. *Let* $\vec{a} \in \mathbb{R}^n$. *The* position vector of $\vec{a}$ *is the directed line segment from the origin* $\vec{0} \in \mathbb{R}^n$ *to the point* $\vec{a}$.

Thus the information of the vector $\vec{a}$ is equivalent to the information of its position vector. This position vector is the geometric realization of $\vec{a}$.

Now we can explore the geometry of addition of vectors. The vector $\vec{a} + \vec{b}$ is, after all, the vector whose coordinates are the sums of coordinates of $\vec{a}$ and $\vec{b}$. This is found geometrically by placing the position vector of $\vec{b}$ at the end of the position vector of $\vec{a}$. Indeed, the result is a vector whose coordinates are the sums of $\vec{a}$ and $\vec{b}$. See Figure 1.

This is called (not surprisingly) the *parallelogram law* of vector addition. We can use the parallelogram law to find the geometric realization of $\vec{a} - \vec{b}$. Indeed, since

$$\vec{a} - \vec{b} = \vec{a} + (-\vec{b}),$$

we construct the corresponding parallelogram for $\vec{a}$ and $-\vec{b}$. Note that $-\vec{b}$ is simply the reflection of $\vec{b}$ through the origin. The resulting vector is shown in Figure 1.

Remark 4. It is worth pointing out (no pun intended) that the vector $\vec{a} - \vec{b}$ points from $\vec{b}$ to $\vec{a}$.

With this in mind, we make the following definition.

Definition 5. *Let* $p_1 = (x_1, y_1, z_1)$ *and* $p_2 = (x_2, y_2, z_2)$ *be points in* $\mathbb{R}^3$. *The* displacement vector from p_1 to p_2 *is*

$$\overrightarrow{p_1 p_2} = p_2 - p_1 = (x_2 - x_1, y_2 - y_1, z_2 - z_1).$$

1.2. **Bases.** The last notion from linear algebra that is of immediate need is the notion of *basis* of a vector space.

Definition 6. *A* basis *of* $\mathbb{R}^n$ *is a collection* B *of* n *vectors*

$$B = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$$

such that any vector $\vec{v} \in \mathbb{R}^n$ *may be uniquely written as a scalar linear combination of the elements of* B, *i.e. there exists a unique choice of numbers* $k_1, \dots, k_n \in \mathrm{RR}$ *such that*

$$\overrightarrow{v} = k_1\vec{b}_1 + k_2\vec{b}_2 + \cdots + k_n\vec{b}_n.$$

Example 7. *The* standard basis *of* $\mathbb{R}^n$ *is the collection*

$$\begin{aligned} \vec{e_1} &= (1, 0, 0, \dots, 0) \\ &\vdots \\ \vec{e_i} &= (0, \dots, 0, 1, 0, \dots, 0) \\ &\vdots \\ \vec{e_n} &= (0, 0, \dots, 0, 1) \end{aligned}$$

Here the i^{th} *vector is zero is all but the* i^{th} *position.*

Example 8. *For* $\mathbb{R}^2$ *and* $\mathbb{R}^3$, *we use even more specialized notation. Let* $i = (1, 0)$ *and* $j = (0, 1)$ *be the standard basis vectors in the plane. We abuse notation by using the same symbols in 3-space: Let* $i = (1, 0, 0)$, $j = (0, 1, 0)$ *and* $k = (0, 0, 1)$ *denote the standard basis of* $\mathbb{R}^3$. *Then, for example,*

$$(1, 0, -\pi) = i - k\pi.$$

2. Lines in space

We now have sufficient background to discuss some basic spacial geometry. Let's study lines? What is a line in space? How can we describe it? What is the equation of a line? Let's study these questions.

First, a line $L \subset \mathbb{R}^3$ in space is uniquely determined by a point $p_0 \in L$ and a vector $\vec{a}$ parallel to L.

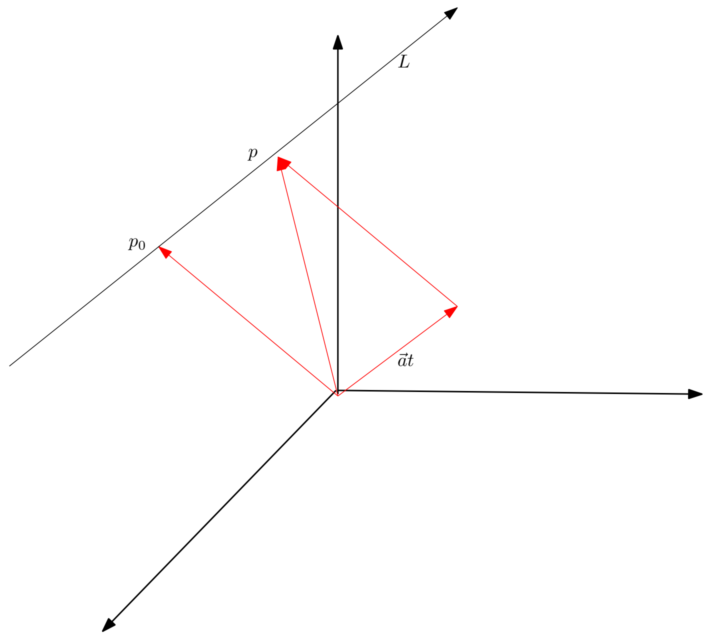

FIGURE 2. Derivation of parametric equation of a line in $\mathbb{R}^3$

This characterization will allow us to write the equation of L. Indeed, suppose we are given $p_0 \in L$ and $\vec{a}$. The equation of L is merely a way to describe the set of points in $\mathbb{R}^3$ which live on L. Let $p \in L$ be any such point. Then the displacement vector from p_0 to p must be parallel to $\vec{a}$ and hence it is a multiple of $\vec{a}$: there exists $t \in \mathbb{R}$ such that

$$\overrightarrow{p_0 p} = t\vec{a}$$

Moreover, the point p is obtained by moving along $t\vec{a}$ from p_0. See Figure 2. By the parallelogram law, we have

$$p = p_0 + t\vec{a}.$$

Thus we have a complete description of L.

$$L = \{p_0 + t\vec{a} : t \in RR\}.$$

Consider the associated map $r(t) : \mathbb{R} \to \mathbb{R}^3$ given by

$$r(t) = p_0 + t\vec{a}. \tag{1}$$

By the above, the image of this map is L. Thus, we have found an equation for L.

2.1. **Parametric equations.** In order to elaborate on this discussion, let's unpack Equation 1 in terms of coordinates. Let

$$\vec{a} = (a_1, a_2, a_3) \qquad\qquad p_0 = (b_1, b_2, b_3).$$

Then

$$r(t) = (a_1 t + b_1, a_2 t + b_2, a_3 t + b_3).$$

But for any $t \in \mathbb{R}$, $r(t) \in \mathbb{R}^3$. Hence $r(t) = (x(t), y(t), z(t))$. Therefore we have

$$\begin{aligned} x(t) &= a_1 t + b_1 \\ y(t) &= a_2 t + b_2 \\ z(t) &= a_3 t + b_3. \end{aligned}$$

Definition 9. *The above equations are called the* parametric equations *for the line* L.

Remark 10. We have shown that a line L in $\mathbb{R}^3$ is the image of a map

$$\begin{aligned} r &: \mathbb{R} \to \mathbb{R}^3 \\ r(t) &= (x(t), y(t), z(t)) \end{aligned}$$

where x, y and z are *linear* functions of t!