LECTURE 15: TRIPLE INTEGRALS AND VOLUME

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1. TRIPLE INTEGRALS AFTER F. SU

Last time we saw an introduction to triple integration – integration over three dimensional regions. The following steps are useful when considering how to find limits of integration.

(1) Chop the region of integration into cubes, which we think of as having little volume (dV).
(2) Stacking these cubes in one direction yields the limits of the innermost integral.
(3) Collapse the region along this chosen direction onto its 2-dimensional shadow.
(4) The limits of integration of the outer integral are determined by this 2D shadow as before.

Example 1. *Find the volume in the first octant bounded by* $x^2 + z^2 = 4$ *and* $y = 3$.

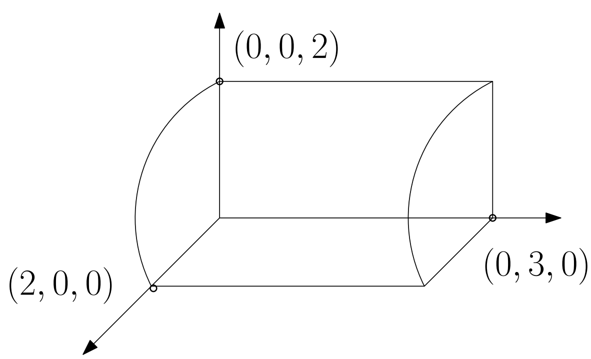

FIGURE 1. Our cylindrical region.

We then have

$$\begin{aligned}
V &= \iiint dV \\
&= \int_{x=0}^{x=2} \int_{y=0}^{y=3} \int_{z=0}^{z=\sqrt{4-x^2}} 1\, dz\, dy\, dx \\
&= \int_{x=0}^{x=2} \int_{z=0}^{z=\sqrt{4-x^2}} \int_{y=0}^{y=3} 1\, dy\, dz\, dx \\
&= \int_{y=0}^{y=3} \int_{z=0}^{z=2} \int_{x=0}^{x=\sqrt{4-z^2}} 1\, dx\, dy\, dz
\end{aligned}$$

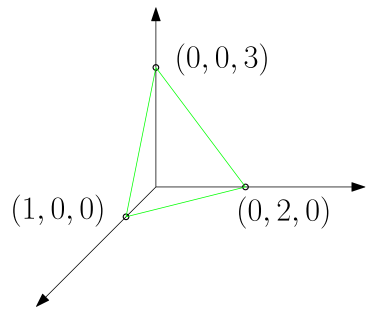

FIGURE 2. Our planar region.

Example 2. *Find the volume in the first octant bounded by* $6x + 3y + 2z = 6$.

Then

$$\begin{aligned} V &= \iiint dV \\ &= \int_{x=0}^{x=1} \int_{y=0}^{y=2-2x} \int_{z=0}^{z=3-3x-3y/2} dz\,dy\,dx. \end{aligned}$$

Example 3. *Find the volume of the region bounded by* $z = y^2$, $y = 0$, $y = -1$, *the* xy*-plane,* $x = 0$ *and* $x = 1$.

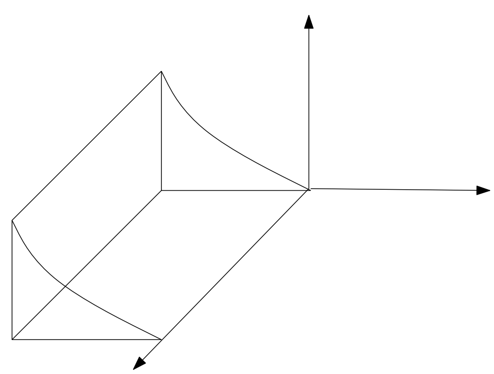

FIGURE 3. Our quarter pipe.

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\begin{aligned}
V &= \iiint dV \\
&= \int_{x=0}^{x=1} \int_{y=-1}^{y=0} \int_{z=0}^{z=y^2} dz\,dx\,dy \\
&= \int_{y=-1}^{y=0} \int_{x=0}^{x=1} \int_{z=0}^{z=y^2} dz\,dx\,dy \\
&= \int_{y=-1}^{y=0} \int_{z=0}^{z=y^2} \int_{x=0}^{x=1} dx\,dz\,dy \\
&= \int_{z=0}^{z=1} \int_{y=-1}^{y=\sqrt{z}} \int_{x=0}^{x=1} dx\,dy\,dz.
\end{aligned}
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Example 4. *The volume of the region between $z = x^2 + y^2$ and $z = 2y$ is given by*

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\begin{aligned}
V &= \iiint dV \\
&= \int_{y=0}^{y=2} \int_{x=-\sqrt{y^2-2y}}^{x=\sqrt{y^2-2y}} \int_{z=x^2+y^2}^{z=2y} dz\,dx\,dy.
\end{aligned}
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