LECTURE 16: CYLINDRICAL AND SPHERICAL COORDINATES

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1. Polar coordinates on $\mathbb{R}^2$

Recall polar coordinates of the plane.

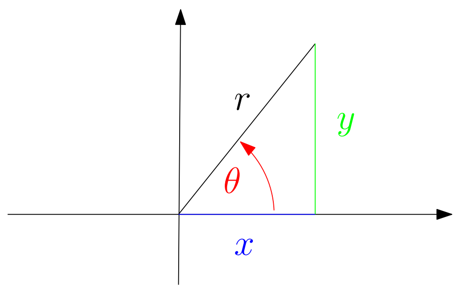

FIGURE 1. Polar coordinates on $\mathbb{R}^2$.

We have

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$

We compute the infinitessimal area (the area form) dA by considering the area of a small section of a circular region in the plane. See Figure 1. Therefore

$$\boxed{dA = r dr d\theta}$$

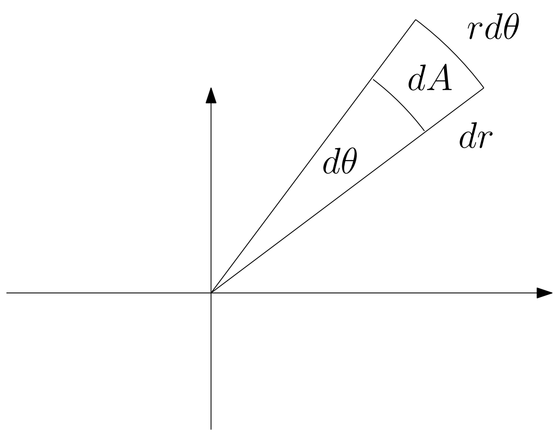

FIGURE 2. The polar form of dA.

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Example 1. *The area of a disk of radius* a *is* πa^2. *Indeed, we compute*

$$\begin{aligned}
A &= \iint_{Disk} dA \\
&= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a} r dr d\theta \\
&= \int_{\theta=0}^{\theta=2\pi} \frac{1}{2} a^2 d\theta \\
&= 2\pi \left(\frac{a^2}{2} \right) \\
&= \pi a^2.
\end{aligned}$$

2. CYLINDRICAL COORDINATES ON $\mathbb{R}^3$

How can we generalize polar coordinates to three dimensions? Maybe the easiest way is to do nothing... Well, polar coordinates already replace x and y, so we can simply do nothing to z. This actually turns out to be rather useful, and this system of coordinates is called *cylindrical*. Can you see why?

Definition 2. Cylindrical coordinates *on* $\mathbb{R}^3$ *are given by* (r, θ, z), *where* (r, θ) *are polar coordinates on the* xy *plane.*

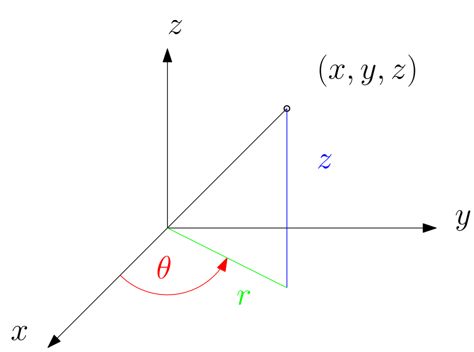

FIGURE 3. Cylindrical coordinates on $\mathbb{R}^3$.

Explicitly, the relation between Cartesian and cylindrical coordinates is given as follows.

$$x = r\cos\theta \qquad z = z \qquad y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

We use the same argument as in polar coordinates to determine the volume form dV.

$$\boxed{dV = r dr d\theta dz}$$

Example 3. *Find the volume of a sphere of radius* R. *We compute*

$$\begin{aligned} V &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=R} \int_{z=-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} r\,dz\,dr\,d\theta \\ &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=R} 2r\sqrt{R^2-r^2}\,dr\,d\theta \\ &= \int_{\theta=0}^{\theta=2\pi} -2/3(R^2-r^2)^{3/2}|_0^R\,d\theta \\ &= 4/3\pi R^3. \end{aligned}$$

3. SPHERICAL COORDINATES ON $\mathbb{R}^3$

We saw that our most naive approach to generalizing polar coordinates lead to cylindrical coordinates. Are there alternative approaches? The geometry of polar coordinates tells us to think of points in the plane differently. If we think of coordinates as a set of directions, telling us how to travel from the origin to a given point, then Cartesian coordinates tell us to think of points in terms of right and left and up and down (x and y). Polar coordinates, on the other hand, tell us to first identify the correct direction (θ) and then head directly there by going the correct distance (r).

The way to generalize this geometry, the idea of finding the correct direction and then heading straight to our destination, is called spherical coordinates.

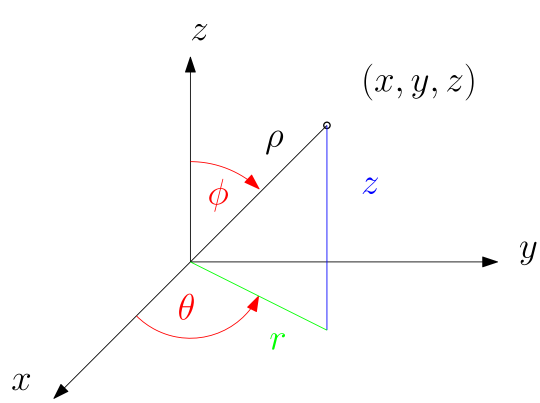

FIGURE 4. Spherical coordinates on $\mathbb{R}^3$.

Let (x, y, z) be a point in Cartesian coordinates in $\mathbb{R}^3$. In spherical coordinates, we use two angles. Let θ be the angle between the x-axis and the position vector of the point $(x, y, 0)$, as before. Now, let $0 \leqslant \phi \leqslant \pi$ be the angle between the positive z-axis and the position vector of (x, y, z). Finally, let ρ be the length of the position vector (x, y, z), i.e. the distance between (x, y, z) and the origin. These are the *spherical coordinates* on $\mathbb{R}^3$. See Figure 3.

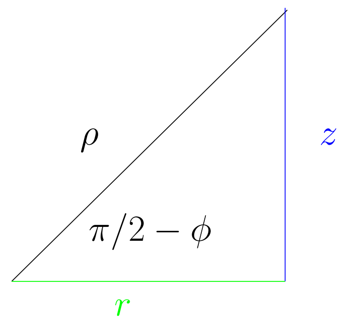

FIGURE 5. Translating between coordinates.

To translate between coordinate systems, consider the right triangle depicted in Figure 3. We have

$$\begin{aligned} r &= \rho\cos(\pi/2 - \phi) \\ &= \rho\left(\cos(\pi/2)\cos\phi + \sin(\pi/2)\sin\phi\right) \\ &= \rho\left(0 + 1\cdot\sin\phi\right) \\ &= \rho\sin\phi \end{aligned}$$

Thus $x = r\cos\theta = \rho\sin\phi\cos\theta$. We compute y and z similarly.

$$\begin{aligned} x &= \rho\sin\phi\cos\theta & \rho^2 &= x^2 + y^2 + z^2 \\ y &= \rho\sin\phi\sin\theta & \tan\phi &= \frac{\sqrt{x^2+y^2}}{z} \\ z &= \rho\cos\phi & \tan\theta &= \frac{y}{x} \end{aligned}$$

To find the volume form dV in spherical coordinates, we consider a small spherical region.

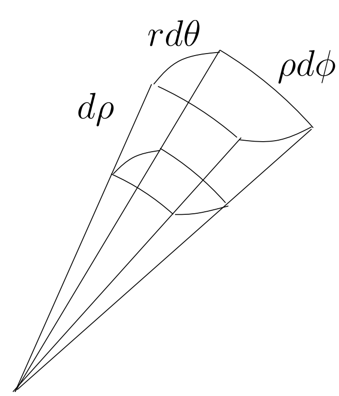

FIGURE 6. The spherical volume form.

Then we compute

$$\begin{aligned} dV &= \rho d\phi \cdot d\rho \cdot r d\theta \\ &= \rho d\phi \cdot d\rho \cdot \rho\sin\phi d\theta \\ &= \rho^2\sin\phi d\rho d\phi d\theta. \end{aligned}$$

Therefore

$$\boxed{dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta}$$

Example 4. *Let's compute the volume of a sphere of radius* R *again.*

$$\begin{aligned}
V &= \iiint_{Sphere} dV \\
&= \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi} \int_{\rho=0}^{\rho=R} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\
&= \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=2\pi} R^3/3 \sin\phi\, d\rho\, d\phi\, d\theta \\
&= \frac{R^3}{3} \int_{\theta=0}^{\theta=2\pi} (-\cos\pi + \cos 0)\, d\theta \\
&= \frac{2R^3}{3} \cdot 2\pi \\
&= \frac{4}{3}\pi R^3.
\end{aligned}$$