LECTURE 7: THE DERIVATIVE OF $F : \mathbb{R}^N \to \mathbb{R}$

DAGAN KARP

1. Limits

In our last lecture, we discussed the tangent plane to the graph of a function $f(x, y)$. The differentiability of f is related to this tangent plane, and a rigorous definition of differentiability requires the notion of limit.

Definition 1. *Let* $f : \mathbb{R}^n \to \mathbb{R}^m$ *be a multivariable function, and* $\mathbf{x} = (x_1, \ldots, x_n)$ *be coordinates of* $\mathbb{R}^n$ *and* $\mathbf{a} \in \mathbb{R}^n$ *be a point. Then* the limit of f as $\mathbf{x}$ approaches $\mathbf{a}$ is equal to L *if, for every* $\epsilon > 0$ *there exists* $\delta > 0$ *such that*

$$|\mathbf{x} - \mathbf{a}| < \delta$$

implies

$$|f(\mathbf{x}) - f(\mathbf{a})| < \epsilon.$$

In that case, we write

$$\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = L.$$

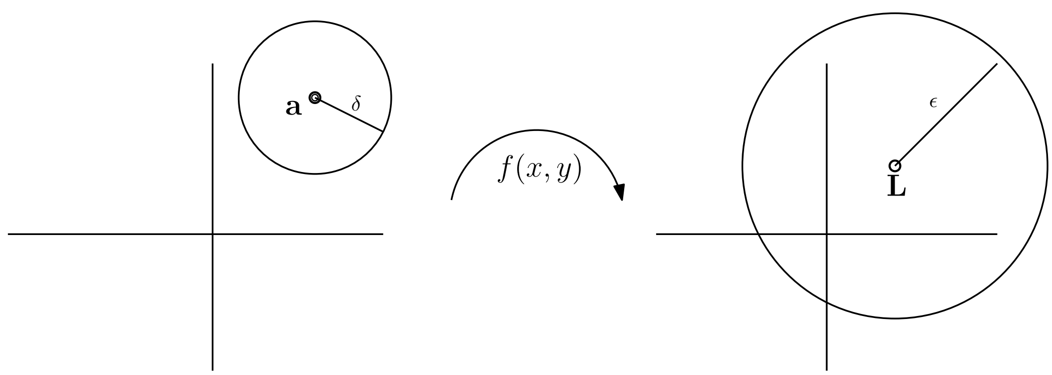

FIGURE 1. The idea of limit.

This definition is directly analogous to the definition of limit for a function of a single variable. Of course, in this higher dimensional version, there are (infinitely) more than two directions to check (as opposed to only left and right hand limits) for a limit to exits.

Also, we have a definition of continuity which directly generalizes the single variable case.

Definition 2. *The function* $f : \mathbb{R}^n \to \mathbb{R}^m$ *is* continuous at $\mathbf{a}$ *if*

$$\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = f(\mathbf{a}).$$

If $U \subset \mathbb{R}^n$ *is a subset and* f *is continuous at* $\mathbf{a}$ *for all* $\mathbf{a} \in U$*, then* f *is* continuous on U.

Date: April 2, 2009.

2. DERIVATIVE

Using our notion of limit, we can define differentiability of our functions of two variables.

Definition 3. *The function* $f : \mathbb{R}^2 \to \mathbb{R}$ *is* differentiable *at* (a, b) *if* f_x *and* f_y *exist and, for*

$$h(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b),$$

we have

$$\lim_{(x,y)\to(a,b)} \frac{f(x, y) - h(x, y)}{|(x, y) - (a, b)|} = 0.$$

Remark 4. We say that $h(x, y)$ is a good linear approximation of $f(x, y)$ at (a, b) in case f is differentiable at (a, b), and the plane $z = h(x, y)$ is the tangent plane to $f(x, y)$ at (a, b).

Example 5. *Let* $f(x, y) = x^2 + y^2$*. What is the tangent plane to* $z = f(x, y)$ *at the point* $(1, 2)$*? We compute*

$$f_x = 2x \qquad\qquad f_y = 2y$$

Thus $f_x(1, 2) = 2$ *and* $f_y(1, 2) = 4$*. Also* $f(1, 2) = 5$*. Thus the tangent plane at* $(1, 2)$ *is*

$$z = h(1, 2) = 5 + 2(x - 1) + 4(y - 2).$$

Example 6. *Let's see that this* h *is in fact a good approximation for* f *in the previous example. We approximate* $f(1.02, 1.95)$*.*

$$(1.02)^2 + (1.95)^2 \sim 5 + 2(.02) + 4(-.05) = 4.84.$$

The actual value is 4.8429*.*

Proposition 7. *The function* $f(x, y)$ *is differentiable at* (a, b) *if* f_x *and* f_y *exist and are continuous in a neighborhood of* (a, b)*.*

Note that we may rewrite h as follows.

$$\begin{aligned} h(x, y) &= f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \\ &= f(a, b) + \left(\frac{\partial f}{\partial x}(a, b), \frac{\partial f}{\partial y}(a, b)\right) \cdot ((x - a), (y - b)). \end{aligned}$$

It is useful to introduce notation for this vector $\left(\frac{\partial f}{\partial x}(a, b), \frac{\partial f}{\partial y}(a, b)\right)$. It is called the *gradient of* f and denoted ∇f. In fact, we can define such an object for functions $\mathbb{R}^n \to \mathbb{R}$.

Definition 8. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a function. The* gradient *of* f *is the vector of partial derivatives*

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

Using this notation, we have the following expression for our tangent plane to the graph of $f(x, y)$ at the point $(a, b, f(a, b))$,

$$h(x, y) = f(a, b) + \nabla f(a, b) \cdot (x - a, y - b).$$

We can now generalize our notion of differentiability to functions on $\mathbb{R}^n$.

Definition 9. *The function* $f : \mathbb{R}^n \to \mathbb{R}$ *is* differentiable *at the point* $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ *if all partial derivatives* $\frac{\partial f}{\partial x_i}$ *exist, for* $i = 1, \ldots, n$ *and, for*

$$h(\mathbf{x}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}),$$

this limit vanishes,

$$\lim_{\mathbf{x} \to \mathbf{a}} \frac{f(\mathbf{x}) - h(\mathbf{x})}{|(\mathbf{x}) - (\mathbf{a})|} = 0.$$

Remark 10. Again, this indicates that the function h is the best linear approximation of f near the point $\mathbf{a}$.